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Summary of Radiation Transport and Radiation Hydrodynamics

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by

B. R. Wienke

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ABSTRACT

The principles of radiative dynamics are summarized and associated physical considerations and approximations are discussed briefly. The classical equations are presented, relativistic effects added and some conventional assumptions and schemes treated.

I. INTRODUCTION

Hydrodynamics and radiative transport are often coupled in analyses of physical problems. Effects of the radiation field are reflected in the hydrodynamic momentum and energy balances when photons interact with the fluid material. Hydrodynamic and thermodynamic factors determine the velocity and temperature of a fluid, which in turn affect the radiative source function, absorption coefficient and scattering kernel in the radiation transport equation. Relativistic corrections to the classical radiative interaction equations (radiation transport + radiation hydrodynamics) must be considered for large fluid velocities.

Ample consideration[1-6] has been given to radiation dynamics. It is the purpose of this work to collect and describe the fundamentals and underlying principles of radiation transport and radiation hydrodynamics in an abbreviated fashion. Section II lists the fundamental equations. Section III considers relativistic effects, and Sec. IV deals with some standard approximation procedures.

II. RADIATION DYNAMICS

The defining equations of radiative dynamics have been developed in a number of references.[1-6] One can summarize them in some inertial frame to which all quantities are referred and measured.

A. Radiation Transport

The radiative transport equation treats photons as point particles of frequency ν moving at the speed of light c in direction $\vec{\Omega}$ with distribution function $f(\vec{r},\nu,\vec{\Omega},t)$ and specific intensity I

$$I(\vec{r},\nu,\vec{\Omega},t) = hc\nu f(\vec{r},\nu,\vec{\Omega},t) \tag{2.1}$$

and takes the form

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I + \sigma_a I = \int d\nu' d\Omega' \frac{\nu}{\nu'} \mu_s' I'(1 + c^2 I/2h\nu^3) - \int d\nu' d\Omega' \mu_s I(1 + c^2 I'/2h\nu'^3) + S(1 + c^2 I/2h\nu^3) \tag{2.2}$$

for the source intensity S, absorption and scattering cross sections σ_a, σ_s, and scattering kernels μ_s, μ_s' defined in a uniform material,

$$S = S(\vec{r},\nu,\vec{\Omega},t)$$

$$I' = I(\vec{r},\nu',\vec{\Omega}',t)$$

$$\sigma_a = \sigma_a(\nu,\vec{\Omega})$$

$$\sigma_s = \sigma_s(\nu,\vec{\Omega})$$

$$\mu_s = \mu_s(\nu, \vec{\Omega} \to \nu', \vec{\Omega}')$$

$$\mu_s' = \mu_s(\nu', \vec{\Omega}' \to \nu, \vec{\Omega}) \tag{2.3}$$

and

$$\sigma_s = \int d\nu' d\Omega' \mu_s(\nu, \vec{\Omega} \to \nu', \vec{\Omega}')$$

$$\sigma_s' = \int d\nu d\Omega \mu_s(\nu', \vec{\Omega}' \to \nu, \vec{\Omega}) \quad . \tag{2.4}$$

Terms involving $c^2I/2h\nu^3$ and $c^2I'/2h\nu'^3$ are of quantum origin and represent enhancement of photon scattering in the presence of other photons in the final state.* The integral term with plus sign in Eq. (2.2) represents in-scattering $(\nu', \vec{\Omega}' \to \nu, \vec{\Omega})$ while the integral term with minus sign denotes out-scattering $(\nu, \vec{\Omega} \to \nu', \vec{\Omega}')$. The presence of the induced scattering terms in the integro-differential Boltzmann equation further renders the equation quadratic in I.

The source S usually denotes photons arising from spontaneous atomic emission[7] and is treated in the local thermodynamic equilibrium (LTE) approximation,*

$$S = \kappa_a B \tag{2.5}$$

with

$$\kappa_a = \sigma_a(1 - e^{-h\nu/kT})$$

$$B = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1} \tag{2.6}$$

for T the temperature in absolute units. In Eq. (2.6) the exponential term in κ_a measures the effective decrease in absorption due to induced emission. At complete thermodynamic equilibrium (CTE – no space or time dependence), the radiation field I must equal the Planck blackbody distribution B and, consequently, the number of photons in-scattered must equal the number out-scattered, yielding a detailed balance condition for the scattering kernels,

$$(1 + c^2B/2h\nu^3)\mu_s' B'/\nu' = (1 + c^2B'/2h\nu'^3)\mu_s B/\nu \quad . \tag{2.7}$$

The Compton scattering kernels appearing in Eqs. (2.2) and (2.3) take the general form,[8,9]

$$\mu_s(\nu', \vec{\Omega}' \to \nu, \vec{\Omega}) = \delta(\varepsilon' + \nu' - \varepsilon - \nu)$$

$$\frac{1}{2} n a^2 \left[\frac{m^2\nu}{\varepsilon(\varepsilon'\nu' - \vec{q}' \cdot \vec{p}')}\right]$$

$$\left[\left(\frac{m^2}{\varepsilon'\nu' - \vec{q}' \cdot \vec{p}'} - \frac{m^2}{\varepsilon'\nu - \vec{q} \cdot \vec{p}'}\right)^2\right.$$

$$+2\left(\frac{m^2}{\varepsilon'\nu' - \vec{q}' \cdot \vec{p}'} - \frac{m^2}{\varepsilon'\nu - \vec{q} \cdot \vec{p}'}\right)$$

---

*If the probability for photon scattering into some final state is P and the number of photons already present in that final state is n, then the overall probability of scattering a photon into the final state is given by P',

$$P' = P(1 + n) \quad .$$

By the uncertainty principle, we know that the phase space for bosons is constrained in measurement by

$$\Delta\vec{p}\Delta\vec{r} \geq h^3/2$$

and consequently, using $d^3p = (h/c)^3\nu^2 d\nu d\Omega$, $I = hc\nu f$,

$$n = \int f\, d^3r d\nu d\Omega = c^2I/2h\nu^3 \quad .$$

Therefore,

$$P' = P(1 + c^2I/2h\nu^3)$$

and the second term in P' is referred to as the induced scattering piece. See, for instance, R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on Physics III (Addison-Wesley Publishing Co., Reading, 1966); J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields (McGraw-Hill Book Co., New York, 1965).

*The local thermodynamic equilibrium approximation assumes that the interaction of the radiation field with the material does not affect, or alter, the local thermodynamic properties of the material.

$$+\frac{\varepsilon'\nu' - \vec{q}'\cdot\vec{p}'}{\varepsilon'\nu - \vec{q}\cdot\vec{p}'} + \frac{\varepsilon'\nu - \vec{q}\cdot\vec{p}'}{\varepsilon'\nu' - \vec{q}'\cdot\vec{p}'}\Bigg] \quad , \tag{2.8}$$

subject to three-momentum conservation

$$\vec{p} = \vec{p}' + \vec{q}' - \vec{q} \quad , \tag{2.9}$$

where $\vec{q}', \vec{q}$ denote the incident and outgoing photon momenta, $\vec{p}', \vec{p}$ the incident and outgoing electron momenta and $\varepsilon', \varepsilon$ the incident and outgoing electron energy, with n the number density of target electrons and the electron radius a defined in terms of the electron rest mass m and charge e,

$$a = \frac{e^2}{mc^2} = 2.8 \times 10^{-13}\ \text{cm} \quad . \tag{2.10}$$

If we assign some arbitrary distribution of electrons $f(\alpha)$ in the particular frame, the effective scattering kernel $\bar{\mu}_s$ is simply obtained by folding μ_s over $f(\alpha)$,

$$\bar{\mu}_s(\nu', \vec{\Omega}' \rightarrow \nu, \vec{\Omega}) = \int d\alpha f \mu_s(\nu', \vec{\Omega}' \rightarrow \nu, \vec{\Omega}) \quad , \tag{2.11}$$

where $d\alpha$ is symbolic for any convenient integration parameter α chosen to describe the distribution.* In the case of stationary electron targets, $\vec{p}' = 0$, $\varepsilon' = m$, Eq. (2.8) reduces to the well-known Klein-Nishina relationship,[10]

$$\mu_s(\nu', \vec{\Omega}' \rightarrow \nu, \vec{\Omega}) = \delta\left(\cos\theta - 1 - \frac{m}{\nu'} + \frac{m}{\nu}\right) \cdot \frac{1}{2} na^2 \left(\frac{m}{\nu'^2}\right)\left[1 + \cos^2\theta + (1 - \cos\theta)^2 \frac{\nu\nu'}{m^2}\right] \tag{2.12}$$

with $\vec{q}' \cdot \vec{q} = qq' \cos\theta$.

*One demands normalization to unity,

$$\int f(\alpha) d\alpha = 1$$

and, of course, some quantitative relationship between α, m, $\vec{p}$, ε, $\vec{q}$, ν, and ν'.

The transport equation [Eq. (2.2)] describes photons moving in a vacuum (index of refraction = 1) and does not account for the two polarization states of light quanta. Furthermore, Eq. (2.2) cannot possibly trace the wave behavior of light. Detailed attention to these points, however, has been given by others[5,7,11-13] and modifications to the transport equation have been suggested.

B. Radiation Hydrodynamics

The term radiation hydrodynamics implies inclusion of a radiation field in the hydrodynamic equations of a fluid. The hydrodynamic equations are essentially conservation statements for particle, momentum and energy in a differential volume element in space. For significant radiation fields, contributions from the radiative momentum and energy must be included in those balances. Furthermore, for large fluid velocities, relativistic corrections, and the attendant Lorentz transformation properties of the radiative and hydrodynamical quantities should be considered. The nonrelativistic equations for an ideal, compressible fluid interacting with a radiation field[4-6,14-16] are easily detailed.

The radiative energy density E^r, radiative energy flux $\vec{F}^r$, and radiative momentum flux $\vec{\vec{P}}^r$ are defined by successive moments of I over $\vec{\Omega}$ (called E, $\vec{F}$ and $\vec{\vec{P}}$),

$$E^r = c^{-1}\int d\nu d\Omega I = c^{-1}\int d\nu E$$

$$\vec{F}^r = \int d\nu d\Omega \vec{\Omega} I = \int d\nu \vec{F}$$

$$\vec{\vec{P}}^r = c^{-1}\int d\nu d\Omega \vec{\Omega}\vec{\Omega} I = c^{-1}\int d\nu \vec{\vec{P}} \quad . \tag{2.13}$$

These are added to the corresponding material terms in the balance equations. Assuming that the macroscopic velocity of the fluid is $\vec{u}$, denoting the mass density, excitation energy density* and pressure of the material ρ_o, t_o and p_o and admitting a source of heat W for the fluid, the radiation hydrodynamic balances consequently take the form in the fixed inertial frame (Eulerian representation)

*The excitation energy density t_o is all energy density in excess of the rest mass energy density $\rho_o c^2$.

$$\frac{\partial \rho_o}{\partial t} + \vec{\nabla} \cdot \rho_o \vec{u} = 0$$

$$\frac{\partial}{\partial t} (\rho_o \vec{u} + \vec{F}^r/c^2) + \vec{\nabla} p_o + \vec{\nabla} \cdot (\rho_o \vec{u}\vec{u} + \vec{\vec{P}}^r) = 0$$

$$\frac{\partial}{\partial t} (\frac{1}{2} \rho_o u^2 + t_o + E^r) + \vec{\nabla} \cdot (\frac{1}{2} \rho_o u^2 + t_o + p_o) \cdot \vec{u} + \vec{\nabla} \cdot \vec{F}^r = W \quad . \tag{2.14}$$

If one introduces the convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \quad , \tag{2.15}$$

Eqs. (2.14) can be recast (Lagrangian representation)

$$\rho_o \frac{D\rho_o^{-1}}{Dt} - \vec{\nabla} \cdot \vec{u} = 0$$

$$\rho_o \frac{D}{Dt} (\vec{u} + \rho_o^{-1} \vec{F}^r/c^2) + \vec{\nabla} p_o + \vec{\nabla} \cdot (\vec{\vec{P}}^r - \vec{u}\, \vec{F}^r/c^2) = 0$$

$$\rho_o \frac{D}{Dt} (\frac{1}{2} u^2 + \rho_o^{-1} t_o + \rho_o^{-1} E^r) + \vec{\nabla} \cdot (\vec{F}^r + p_o \vec{u} - E^r \vec{u}) = W \quad . \tag{2.16}$$

Equations (2.14) are conservative in a fixed volume element in the inertial frame, while Eqs. (2.16) are conservative in a volume element moving with velocity $\vec{u}$ through the frame. In the two sets, Eqs. (2.14) and (2.16), the three equations detail conservation of particles, momentum and energy, respectively. That D/Dt represents the time rate of change in the moving frame is noted from the relationship

$$\int_V d^3r \rho_o \frac{DQ}{Dt} = \frac{\partial}{\partial t} \int_V d^3r \rho_o Q \quad , \tag{2.17}$$

with Q any arbitrary quantity and V a moving volume whose boundary points move with local fluid velocity $\vec{u}$.

III. LORENTZ TRANSFORMATIONS AND RELATIVISTIC EFFECTS

The Lorentz transformation[14] L defined on a four-dimensional space-time manifold is written to connect frames ξ and ξ_o moving with relative velocity v along mutual z axes,

$$\xi = L\xi_o \quad , \tag{3.1}$$

with

$$\xi = (ct,\vec{x},\vec{y},\vec{z}) = (ct,\vec{r})$$

$$\xi_o = (ct_o,\vec{x}_o,\vec{y}_o,\vec{z}_o) = (ct_o,\vec{r}_o) \quad , \tag{3.2}$$

x, y, z, x_o, y_o and z_o space coordinates, t and t_o time coordinates, and

$$L = \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \tag{3.3}$$

for

$$\tanh\phi = \beta = v/c$$

$$\cosh\phi = \gamma = (1 - \beta^2)^{-\frac{1}{2}} \quad . \tag{3.4}$$

The frame ξ moves with speed v with respect to ξ_o. The metric tensor on the manifold is defined

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{3.5}$$

with associated invariant inner product,

$$\xi \cdot \xi_o = \xi_i g^{ij} \xi_j = tt_o - \vec{r} \cdot \vec{r}_o \quad . \tag{3.6}$$

The equations of physics are invariant under Lorentz transformation. In other words, the equations are form invariant in arbitrary, nonaccelerated frames provided all quantities appearing in the equations are defined and are measured in the frame. In many applications, quantities are known in one Lorentz frame. It is then necessary to transform both the quantities and their explicit variables to the Lorentz frame of interest. In radiative transfer work, the material source B, absorption coefficient σ_a, and electron density ρ_o are described in the fluid rest frame. The corresponding laboratory quantities are then obtained following Lorentz transformation.

A rule of thumb for determining whether pertinent quantities need be Lorentz transformed, or can be treated classically, is roughly furnished by the categorization,

$$\begin{aligned} v/c &< 1/10 \quad \text{(classical region)} \\ v/c &> 1/10 \quad \text{(relativistic region)} \\ v/c &> 99/100 \quad \text{(extreme relativistic region)} \quad . \end{aligned} \tag{3.7}$$

Use of the classical expressions for momentum and energy in the case $v/c < 1/10$ results in less than 1% error from the fully relativistic predictions. Quantities transforming in powers γ^n might then be expected to deviate less than n% from the relativistic prediction at $v/c = 1/10$, using a linear extrapolation. At the extreme relativistic limit $v/c > 99/100$, the rest energy may be safely neglected to 1% in the calculation.

Assuming the target material moves with velocity $\vec{u}$ and denoting the material rest frame with a subscript zero, as before, we can list the specific Lorentz transformation properties of various quantities after defining

$$\begin{aligned} \Lambda &= (1 - u^2/c^2)^{-\frac{1}{2}} \\ D &= (1 - \vec{\Omega} \cdot \vec{u}/c) \\ D' &= (1 - \vec{\Omega}' \cdot \vec{u}/c) \quad . \end{aligned} \tag{3.8}$$

Noting that the momentum distribution function for both photons and electrons is a Lorentz invariant (e denotes electrons, γ denotes photons), that is,

$$\psi^{e,\gamma}(\vec{p})\, d^3p d^3r = \psi_o^{e,\gamma}(\vec{p}_o) d^3p_o d^3r_o \quad , \tag{3.9}$$

which implies

$$\psi^{e,\gamma}(\vec{p}) = \psi_o^{e,\gamma}(\vec{p}_o) \quad , \tag{3.10}$$

it can be shown that the various terms in the transport equation Lorentz transform[4,5] as,

$$\begin{aligned} \nu &= (\Lambda D)^{-1}\nu_o \\ \nu' &= (\Lambda D')^{-1}\nu'_o \\ \vec{\Omega} &= (\Lambda D)^{-1}\,[\vec{\Omega}_o + \Lambda(\Lambda + 1)(\Lambda D + 1)\vec{u}/c] \\ \vec{\Omega}' &= (\Lambda D')^{-1}[\vec{\Omega}'_o + \Lambda(\Lambda + 1)(\Lambda D' + 1)\vec{u}/c] \\ \frac{1}{c}\frac{\partial}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} &= (\Lambda D)\ \frac{1}{c}\frac{\partial}{\partial t_o} + \vec{\Omega}_o\cdot\vec{\nabla}_o \\ I &= (\Lambda D)^{-3} I_o \\ I' &= (\Lambda D')^{-3} I'_o \\ S &= (\Lambda D)^{-2} S_o \\ \sigma_a &= (\Lambda D)\sigma_{ao} \\ \sigma_s &= (\Lambda D)\sigma_{so} \\ \mu'_s &= (D')^{-1} D\mu'_{so} \\ \mu_s &= (D')^{-1} D\mu_{so} \\ d\Omega' &= (\Lambda D')^2 d\Omega'_o \\ d\Omega &= (\Lambda D)^2 d\Omega_o \\ 1 - \vec{\Omega}\cdot\vec{\Omega}' &= (\Lambda^2 DD')(1 - \vec{\Omega}_o\cdot\vec{\Omega}'_o) \end{aligned} \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (2.2) yields the transport equation in the zero frame (identical to Eq. (2.2) with subscripts zero on all quantities).

Relativistic hydrodynamics is concerned with transformation of fluid rest frame quantities to the frame in which the fluid equations are posed and defined.* In radiative transfer work, effects of a moving medium are seen directly in changes in the electron density and subsequently in the material

*We are not interested in transformation of the hydrodynamic equations from the laboratory frame to any other arbitrary Lorentz frame. A priori, hydrodynamics for an incompressible fluid presupposes two Lorentz frames (spectator and fluid frame). Discussion of relativistic effects is confined to these two frames. Of course, Lorentz transformation of the hydrodynamic equations to another inertial frame (a <u>third</u> frame) is possible, and the usual form invariance of the equations is the expected result.

momentum and energy densities and fluxes. In the case of a compressible fluid ($\vec{\nabla} \cdot \vec{u} \neq 0$), an additional complication arises since the fluid frame is not strictly a Lorentz frame. However, this fact only causes trouble in the attempted Lorentz transformation of derivative operators (e.g., time derivatives, streaming terms). Smooth material functions such as particle, energy and momentum densities and fluxes are thought to evolve continuously and infinitesimally from their local fluid frames.[17] Accordingly then, the various material densities and fluxes as viewed in the Lorentz frame through which the material moves with velocity $\vec{u}$ are given in terms of ρ_o, t_o, p_o. Ultimately, the relativistic hydrodynamic equations take the form[4,5] in the fixed Eulerian picture

$$\frac{\partial}{\partial t}(\Lambda\rho_o) + \vec{\nabla} \cdot \Lambda\rho_o\vec{u} = 0$$

$$\frac{\partial}{\partial t}[\Lambda^2(\rho_o + t_o/c^2 + p_o/c^2)\vec{u} + \vec{F}^r/c^2] + \vec{\nabla}p_o + \vec{\nabla}\cdot[\Lambda^2(\rho_o + t_o/c^2 + p_o/c^2)\vec{u}\vec{u} + \vec{\vec{P}}^r] = 0$$

$$\frac{\partial}{\partial t}[\Lambda^2(\rho_o c^2 + t_o + p_o) - p_o + E^r] + \vec{\nabla}\cdot[\Lambda^2(\rho_o c^2 + t_o + p_o)\vec{u} + \vec{F}^r] = W \tag{3.12}$$

or, in the Lagrangian representation,

$$\Lambda\rho_o \frac{D}{Dt}(\Lambda\rho_o)^{-1} - \vec{\nabla} \cdot \vec{u} = 0$$

$$\Lambda\rho_o \frac{D}{Dt}\left[\frac{\Lambda^2(\rho_o c^2 + t_o + p_o)\vec{u} + \vec{F}^r}{c^2\Lambda\rho_o}\right] + \vec{\nabla}p_o - \vec{\nabla}\cdot(\vec{\vec{P}}^r - \vec{u}\vec{F}^r/c^2) = 0$$

$$\Lambda\rho_o \frac{D}{Dt}\left[\frac{\Lambda\rho_o c^2(\Lambda - 1) + \Lambda^2(t_o + p_o) - p_o + E^r}{\Lambda\rho_o}\right] + \vec{\nabla}\cdot\left[\vec{F}^r + (t_o - E^r)\vec{u}\right] = W \quad . \tag{3.13}$$

In the limit $c \to \infty$, $\Lambda \to 1$, $(\Lambda - 1) \to \frac{1}{2}u^2/c^2$ in the <u>material</u> terms, Eqs. (3.12) and (3.13) reduce to their nonrelativistic expressions, Eq. (2.14) and (2.16), respectively. Terms appearing on the right of $\partial/\partial t$ in Eqs. (3.12) and (3.13) are the total particle, momentum and energy densities as measured in the Lorentz frame. The corresponding total particle, momentum and energy fluxes follow $\vec{\nabla}$ in the above same equations.

The radiative transfer equation can also be posed in the Lagrangian picture by substituting the convective derivative, Eq. (2.15), in the intensity time derivative and using the continuity equation, Eq. (2.16). The left-hand side of Eq. (2.2) yields

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega} \cdot \vec{\nabla}I = \frac{\rho}{c}\frac{D}{Dt}(\rho^{-1}I) + \vec{\nabla}\cdot(\vec{\Omega}I - \frac{\vec{u}}{c}I) \tag{3.14}$$

with $\rho = \Lambda\rho_o$.

IV. APPROXIMATIONS

The equations given in the preceding sections are exact within the framework of special relativity and the classical description of the hydrodynamics of an ideal fluid and the transport of radiative energy interacting with fluid. Further implementation schemes have been employed in effecting solutions.

In the transport equation, Eq. (2.2), the absorption coefficient σ_a and source function $S = \kappa_a B$ are obtained from the corresponding fluid frame quantities using Eqs. (3.11). The prescription employed is to expand the source and absorption terms in powers of u/c and work to low order. It is evident that the source-absorption quantity can be rewritten with the aid of Eqs. (2.5), (2.6) and (3.11),

$$S(1 + c^2I/2h\nu^3) - \sigma_a I = \kappa_a(B - I) = \Lambda D\kappa_{ao}(B_o/\Lambda^3D^3 - I) \tag{4.1}$$

which, to order u/c, gives in a power series expansion

$$\Lambda D\kappa_{ao}(B_o/\Lambda^3D^3 - I) \simeq (1 - \vec{u}\cdot\vec{\Omega}/c)\left\{\left[\kappa_{ao}(\nu) + \frac{\partial\kappa_{ao}}{\partial\nu_o}\bigg|_{\nu_o=\nu}(\nu_o - \nu) + \ldots\right] \times\right.$$

$$(1 + 3\vec{u} \cdot \vec{\Omega}/c) \left[B_o(\nu) + \frac{\partial B_o}{\partial \nu_o}\bigg|_{\nu_o=\nu} (\nu_o - \nu) + \ldots \right] - I \Bigg\} \quad (4.2)$$

The power series in κ_{ao} and B_o effectively changes variable from ν_o to ν as required. Using the Doppler shift relationship from Eqs. (3.11),

$$\nu_o = \Lambda D \nu \quad , \qquad (4.3)$$

it follows to the same order in u/c,

$$\nu_o - \nu \cong (1 - \vec{u} \cdot \vec{\Omega}/c)\nu - \nu = -\nu\vec{u} \cdot \vec{\Omega}/c \quad , (4.4)$$

which allows elimination of the frequency shift variables in Eq. (4.2). Equations (4.1), (4.2) and (4.4) can then be substituted into Eq. (2.2) with, from Eq. (2.6),

$$\nu \frac{\partial B_o}{\partial \nu_o}\bigg|_{\nu_o=\nu} = 3B_o(\nu) - B_o(\nu) \frac{h\nu}{kT(1 - e^{-h\nu/kT})} \quad . \qquad (4.5)$$

Treatments of $\partial \kappa_{ao}/\partial \nu_o \big|_{\nu_o=\nu}$ are model and data dependent and vary.[18,19]

Additionally, the transport equation can be treated in a diffusion-like approximation.[20-22] Multiplying Eq. (2.2) by 1 and $\vec{\Omega}$ and integrating over $d\Omega$ generates the zeroth and first moment equations. A coupled statement is the two term expansion[20] of the intensity (Eddington approximation)*

$$I = \frac{1}{4\pi} E + \frac{3}{4\pi} \vec{\Omega} \cdot \vec{F} \quad . \qquad (4.6)$$

The coupled equations which result upon successive integrations over $d\Omega$ take form, recalling Eqs. (2.13) and expanding the scattering kernels in partial waves,

$$\frac{1}{c}\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = \kappa_a(4\pi B - E) - \sigma_s E + \int d\nu' \frac{\nu}{\nu'} \mu_s^{o'} E'$$

$$+ \frac{c^2}{8\pi h} E \int d\nu' \left(\frac{\mu_s^{o'}}{\nu^2 \nu'} - \frac{\mu_s^o}{\nu'^3} \right) E'$$

$$+ \frac{3c^2}{8\pi h} \vec{F} \cdot \int d\nu' \left(\frac{\mu_s^{1'}}{\nu^2 \nu'} - \frac{\mu_s^1}{\nu'^3} \right) \vec{F}'$$

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \frac{1}{3}\vec{\nabla} E = -(\kappa_a + \sigma_s)\vec{F} + \int d\nu' \frac{\nu}{\nu'} \mu_s^{1'} \vec{F}'$$

$$+ \frac{c^2}{8\pi h} E \int d\nu' \left(\frac{\mu_s^{1'}}{\nu^2 \nu'} - \frac{\mu_s^1}{\nu'^3} \right) \vec{F}'$$

$$+ \frac{c^2}{8\pi h} \vec{F} \int d\nu' \left(\frac{\mu_s^{o'}}{\nu^2 \nu'} - \frac{\mu_s^o}{\nu'^3} \right) E'$$

$$(4.7)$$

when, $d\Omega = 2\pi dz$,

$$\mu_s^\ell = 2\pi \int dz \mu_s P_\ell(z) \quad . \qquad (4.8)$$

In the case of anisotropic elastic scattering[23]

$$\mu_s^\ell = a^\ell \delta(\nu - \nu') \qquad (4.9)$$

with the various a^ℓ constants, the induced scattering terms in Eqs. (4.7) vanish. Requiring Fick's law[24]

$$\vec{F} = -D\vec{\nabla} E \qquad (4.10)$$

yields the diffusion equation with source from the first of Eqs. (4.7), while the second of Eqs. (4.7) defines the diffusion coefficient,[25]

$$D \cong (3\kappa_a + 3\sigma_s - 3a^1)^{-1} \quad . \qquad (4.11)$$

Asymptotic diffusion theory[26] employs the first of Eqs. (4.7), but uses

$$D \cong \frac{1 - s}{\alpha^2(\kappa_a + \sigma_s)} \qquad (4.12)$$

with the scattering probability s defined,

$$s = \frac{\sigma_s}{(\kappa_a + \sigma_s)} \qquad (4.13)$$

and

$$\frac{1}{2\alpha} \ln \frac{1 + \alpha}{1 - \alpha} = s \quad . \qquad (4.14)$$

*The Eddington approximation is also written in generality $\vec{\nabla} \cdot \overleftrightarrow{P} = \vec{\nabla} fE$ with f some function approaching 1/3 in the diffusion limit and 1 in the streaming limit.

Equilibrium diffusion theory[7,27] treats the material as being in complete thermodynamic equilibrium ($B = I$), and uses,

$$D \simeq \frac{\int d\nu (3\kappa_a + 3\sigma_s - 3a^1)^{-1} \frac{\partial B}{\partial T}}{\int d\nu \frac{\partial B}{\partial T}} \tag{4.15}$$

with the same source function B given in Eq. (2.6). Other variations of the above approaches are also employed.[28]

In the Lagrangian picture, the left-hand sides of Eqs. (4.7) become ($\rho = \Lambda\rho_o$),

$$\frac{1}{c}\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = \frac{\rho}{c}\frac{D}{Dt}(\rho^{-1}E) + \vec{\nabla}\cdot(\vec{F} - \frac{\vec{u}}{c}E)$$

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \frac{1}{3}\vec{\nabla}E = \frac{\rho}{c}\frac{D}{Dt}(\rho^{-1}F) + \frac{1}{3}\vec{\nabla}E - \vec{\nabla}\cdot\frac{\vec{u}\vec{F}}{c} \quad . \tag{4.16}$$

Use of the Lagrangian picture in the transport equation allows one to incorporate effects of material changes directly and permits coupling to the Lagrangian hydrodynamical equations.[29,30]

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